

Weighted estimates for commutators of some singular integrals related to Schrödinger operators

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Abstract

Let $L = -\Delta + V$ with non-negative potential V satisfying some appropriate reverse Hölder inequality. In this paper, we study the boundedness of the commutators of some singular integrals associated to L such as Riesz transforms and fractional integrals with the new BMO functions introduced in [BHS1] on the weighted spaces $L^p(w)$ where w belongs to the new classes of weights introduced by [BHS2].

1 Introduction

Let $L = -\Delta + V$ be the Schrödinger operators on \mathbb{R}^n with $n \geq 3$ where the potential V is in the reverse Hölder class RH_q for some $q > n/2$, i.e., V satisfies the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B V(y)^q dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) dy$$

for all ball $B \subset \mathbb{R}^n$.

In this paper, we consider the following singular integrals associated to L :

- (i) Riesz transforms $R = \nabla L^{-1/2}$ and their adjoint $R^* = L^{-1/2} \nabla$;
- (ii) Fractional integrals $I_\alpha f(x) = L^{-\alpha/2} f(x)$ for $0 < \alpha < n$.

In the classical case when $V = 0$, it has been shown that Riesz transforms R and their commutators R_b with BMO functions b is bounded on $L^p(w)$ for all $1 < p < \infty$ and w in the Muckenhoupt classes A_p , see for example [St]. Also, the classical fractional integrals and their commutators with BMO functions b are bounded from $L^p(w^p)$ to $L^q(w^q)$ for all $1 < p < n/\alpha$, $1/p - 1/q = \alpha/n$ and $w \in A_{1+1/p'} \cap RH_q$, or equivalently $w^q \in A_{1+\frac{q}{p'}}$, where A_p is the Muckenhoupt class of weights, see for example [MW, ST]. Recall that

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a non-negative and locally integrable function w is said to be in the Muckenhoupt A_p classes with $1 \leq p < \infty$, if the following inequality holds for all balls $B \subset \mathbb{R}^n$

$$\left(\int_B w \right)^{1/p} \left(\int_B w^{-\frac{1}{p-1}} \right)^{1/p'} \leq C|B|. \quad (1)$$

Recently, in [BHS2], a new class of weights associated to Schrödinger operators L has been introduced. According to [BHS2], the authors defined the new classes of weights $A_p^L = \cup_{\theta>0} A_p^{L,\theta}$ for $p \geq 1$, where $A_p^{L,\theta}$ is the set of those weights satisfying

$$\left(\int_B w \right)^{1/p} \left(\int_B w^{-\frac{1}{p-1}} \right)^{1/p'} \leq C|B| \left(1 + \frac{r}{\rho(x)} \right)^\theta \quad (2)$$

for all ball $B = B(x, r)$. We denote $A_\infty^L = \cup_{p \geq 1} A_p^L$ where the critical radius function $\rho(\cdot)$ is defined by

$$\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V \leq 1 \right\}, \quad x \in \mathbb{R}^n, \quad (3)$$

see [Sh].

It is easy to see that in certain circumstances the new class A_p^L is larger than the Muckenhoupt class A_p . The following properties hold for the new classes A_p^L , see [BHS2, Proposition 5].

Proposition 1.1 *The following statements hold:*

- i) $A_p^L \subset A_q^L$ for $1 \leq p \leq q < \infty$.
- ii) If $w \in A_p^L$ with $p > 1$ then there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}^L$. Consequently, $A_p^L = \cup_{q < p} A_q^L$.

For the new classes A_p^L , the weighted norm inequalities for the some singular integrals associated to L was investigated.

Theorem 1.2 (a) *Let $V \in RH_q$.*

(i) *If $n/2 < q < n$ and s is such that $1/s = 1/q - 1/n$, the Riesz transforms R^* are bounded on $L^p(w)$ for $s' < p < \infty$ and $w \in A_{p/s'}^L$ and hence by duality R is bounded on $L^p(w)$ for $1 < p < s$ with $w^{-\frac{1}{p-1}} \in A_{p'/s'}^L$.*

(ii) *If $q \geq n$, the Riesz transforms R^* and R are bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p^L$.*

(b) *Let $V \in RH_q$ with $q > n/2$. Then I_α is bounded from $L^p(w^p)$ to $L^q(w^q)$ for all $1 < p < n/\alpha$, $1/p - 1/q = \alpha/n$ and $w^q \in A_{1+\frac{q}{p}}^L$.*

For the proof we refer to Theorem 3 and Theorem 4 in [BHS2].

Now we consider the commutators of the Riesz transforms R and R^* with the BMO functions b . It was proved in [GLP] that the commutators R_b and R_b^* are bounded on L^p

here the range of p depends on the potential V . Then the authors in [BHS1] extended the classes of BMO functions to the new class BMO_L^θ with $\theta > 0$ for the boundedness of the commutators R_b and R_b^* . We would like to give a brief overview of the results in [BHS1]. According to [BHS1], the new BMO space BMO_L^θ with $\theta > 0$ is defined as a set of all locally integrable functions b satisfying

$$\frac{1}{|B|} \int_B |b(y) - b_B| dy \leq C \left(1 + \frac{r}{\rho(x)}\right)^\theta \quad (4)$$

where $B = B(x, r)$ and $b_B = \frac{1}{|B|} \int_B b$. A norm for $b \in BMO_L^\theta$, denoted by $\|b\|_\theta$, is given by the infimum of the constants satisfying (4). Clearly $BMO_L^{\theta_1} \subset BMO_L^{\theta_2}$ for $\theta_1 \leq \theta_2$ and $BMO_L^0 = BMO$. We define $BMO_L^\infty = \cup_{\theta > 0} BMO_L^\theta$.

The following result can be considered to be a variant of John-Nirenberg inequality for the spaces BMO_L^θ , see [BHS1, pp.119-120].

Proposition 1.3 *Let $\theta > 0, s \geq 1$. If $b \in BMO_L^\theta$ then for all $B = (x_0, r)$*

i)

$$\left(\frac{1}{|B|} \int_B |b(y) - b_B|^s dx \right)^{1/s} \lesssim \|b\|_\theta \left(1 + \frac{r}{\rho(x_0)}\right)^{\theta'}$$

where $\theta' = (N_0 + 1)\theta$ and N_0 is a constant in (5).

ii)

$$\left(\frac{1}{|2^k B|} \int_{2^k B} |b(y) - b_B| dx \right)^{1/s} \lesssim \|b\|_\theta k \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{\theta'}$$

for all $k \in \mathbb{N}$.

Let T be either R or R^* . For $b \in BMO_L^\infty$ we consider the commutators

$$T_b f(x) = T(bf)(x) - b(x)Tf(x)$$

and

$$I_\alpha^b f(x) = I_\alpha(bf)(x) - b(x)I_\alpha f(x).$$

It was proved in [BHS1, Theorem 1] that

Theorem 1.4 *Let $b \in BMO_L^\theta$ with $\theta > 0$ and $V \in RH_q$.*

(i) If $n/2 < q < n$ and s is such that $1/s = 1/q - 1/n$, the commutators R_b^ are bounded on L^p for $s' < p < \infty$ and hence by duality R_b is bounded on L^p for $1 < p < s$.*

(ii) If $q \geq n$, the commutators R_b^ and R_b are bounded on L^p for $1 < p < \infty$.*

The aim of this paper is investigating the boundedness of the commutators R_b^* , R_b and I_α^b with $b \in BMO_L^\infty$ on the new weighted spaces $L^p(w)$ with $w \in A_p^L$.

The organization of this paper is as follows. In Section 2, we recall some basic properties of the critical radius function $\rho(\cdot)$ and consider weighted estimates for some localized operators. Section 3 is devoted to prove the main results on weighted estimates of the commutators R_b^* , R_b and I_α^b .

Finally, we make some conventions. Throughout the whole paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $X \lesssim Y$ means that there exists a positive constant C such that $X \leq CY$.

Recently, we have learned that the A_p^L weighted norm inequalities for the commutators of the Riesz transforms was obtained independently in [BHS3]. However, the approach in our paper is different from that in [BHS3]. Moreover, the weighted norm inequalities for the commutators of fractional integrals $L^{-\alpha/2}$ is unique.

2 Weighted estimates for some localized operators

We would like to recall some important properties concerning the critical radius function which will play an important role to obtain the main results in the sequel, see [Sh, DZ1] respectively.

Proposition 2.1 *If $V \in RH_{n/2}$, there exist c_0 and $N_0 \geq 1$ such that*

$$c_0^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N_0} \leq \rho(y) \leq c_0\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{N_0}{N_0+1}} \quad (5)$$

for all $x, y \in \mathbb{R}^n$.

A ball of the form $B(x, \rho(x))$ is called a *critical ball*. From the inequality (5), we can imply that for $x, y \in \sigma Q$ where Q is a critical ball and $\sigma > 0$, then

$$\rho(x) \leq c_\sigma \rho(y) \quad (6)$$

where $c_\sigma = c_0^2(1 + \sigma)^{\frac{2N_0+1}{N_0+1}}$.

Proposition 2.2 *There exists a sequence of points $x_j, j \geq 1$ in \mathbb{R}^n so that the family $Q_j := B(x_j, \rho(x_j))$ satisfies*

- (i) $\cup_j Q_j = \mathbb{R}^n$.
- (ii) For every $\sigma \geq 1$ there exist constants C and N_1 such that $\sum_j \chi_{\sigma Q_j} \leq C\sigma^{N_1}$.

Following [BHS1], we introduce the following maximal functions for $g \in L_{\text{loc}}^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$M_{\rho, \alpha} g(x) = \sup_{x \in B \in \mathcal{B}_{\rho, \alpha}} \frac{1}{|B|} \int_B |g|,$$

$$M_{\rho, \alpha}^\# g(x) = \sup_{x \in B \in \mathcal{B}_{\rho, \alpha}} \frac{1}{|B|} \int_B |g - g_B|,$$

where $\mathcal{B}_{\rho, \alpha} = \{B(y, r) : y \in \mathbb{R}^n \text{ and } r \leq \alpha\rho(y)\}$.

Also, given a ball Q , we define the following maximal functions for $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in Q$

$$M_Q g(x) = \sup_{x \in B \in \mathcal{F}(Q)} \frac{1}{|B \cap Q|} \int_{B \cap Q} |g|,$$

$$M_Q^\# g(x) = \sup_{x \in B \in \mathcal{F}(Q)} \frac{1}{|B \cap Q|} \int_{B \cap Q} |g - g_{B \cap Q}|,$$

where $\mathcal{F}(Q) = \{B(y, r) : y \in Q, r > 0\}$.

We have the following lemma.

Lemma 2.3 *For $1 < p < \infty$, then there exist β and γ such that if $\{Q_k\}_k$ is a sequence of balls as in Proposition 2.2 then*

$$\int_{\mathbb{R}^n} |M_{\rho, \beta} g(x)|^p w(x) dx \lesssim \int_{\mathbb{R}^n} |M_{\rho, \gamma}^\# g(x)|^p w(x) dx + \sum_k w(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |g| \right)^p$$

for all $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $w \in A_\infty^L$.

Proof: Note that the unweighted estimate of Lemma 2.3 was obtained [BHS1, Lemma 2], and the weighted estimate was obtained in [B] for the particular case $\rho = 1$. Now we adapt some ideas in [BHS1, Lemma 2] (see also [B]) to our present setting.

According to [BHS1, p. 121], there exists $\beta > 0$ so that for all critical balls Q and $x \in Q$, we have

$$M_{\rho, \beta} g(x) \leq M_{2Q}(g\chi_{2Q})(x),$$

and for $x \in 2Q$,

$$M_{2Q}^\#(g\chi_{2Q})(x) \leq M_{\rho, 2}^\# g(x).$$

Therefore, by the similar argument to that in [B, Lemma 2.4] we obtain for $w \in A_\infty^L$,

$$\begin{aligned} \int_{\mathbb{R}^n} |M_{\rho, \beta} g(x)|^p w(x) dx &\leq \sum_k \int_{Q_k} |M_{\rho, \beta} g(x)|^p w(x) dx \\ &\leq \sum_k \int_{Q_k} |M_{2Q}(g\chi_{2Q})(x)|^p w(x) dx \\ &\lesssim \sum_k \int_{2Q_k} |M_{2Q_k}^\# g(x)|^p w(x) dx + \sum_k w(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |g| \right)^p \\ &\lesssim \sum_k \int_{2Q_k} |M_{\rho, 2}^\# g(x)|^p w(x) dx + \sum_k w(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |g| \right)^p \\ &\lesssim \int_{\mathbb{R}^n} |M_{\rho, 2}^\# g(x)|^p w(x) dx + \sum_k w(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |g| \right)^p. \end{aligned}$$

This completes our proof.

Throughout this paper, we always assume that N is a sufficiently large number and different from line to line. For $\kappa \geq 1, 0 < \alpha < n$ and $1 \leq s < n/\alpha$, we define the following functions for $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$G_{\kappa,\alpha,s}f(x) = \sup_{Q \ni x; Q \text{ is critical}} \sum_{k=0}^{\infty} 2^{-Nk} \left(\frac{1}{|2^k \widehat{Q}|^{1-\alpha s/n}} \int_{2^k \widehat{Q}} |f(z)|^s dz \right)^{1/s}$$

and

$$H_{\kappa,s}f(x) = \sup_{Q \ni x; Q \text{ is critical}} \sum_{k=0}^{\infty} 2^{-Nk} \left(\frac{1}{|2^k \widehat{Q}|} \int_{2^k \widehat{Q}} |f(z)|^s dz \right)^{1/s}$$

where $\widehat{Q} = \kappa Q$.

When $\kappa = 1$, we write $G_{\alpha,s}$ and H_s instead of $G_{1,\alpha,s}$ and $H_{1,s}$, respectively. We are now in position to establish the weighted estimates for $G_{\kappa,\alpha,s}$ and $H_{\kappa,s}$.

Proposition 2.4 (i) Let $w^q \in A^L_{1+\frac{q/s}{(p/s)'}}$. If $p > s$ and $1/p - 1/q = \alpha/n$, then we have

$$\|G_{\kappa,\alpha,s}f\|_{L^q(w^q)} \lesssim \|f\|_{L^p(w^p)}.$$

(ii) Let $w \in A^L_{p/s}$. If $p > s$, then we have

$$\|H_{\kappa,s}f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

Proof:

(i) Without loss of generality, we can assume that $\kappa = 1$. Assume that $Q = B(x_0, \rho(x_0))$. For $x \in Q$, the inequalities (5) tells us that

$$C_0^{-1}\rho(x_0) \leq \rho(x) \leq C_0\rho(x_0).$$

This implies $|B(x, \rho(x))| \approx |Q|$ and $Q \subset 2C_0B(x, \rho(x))$. Therefore,

$$G_{\alpha,s}f(x) \lesssim \sum_{k=0}^{\infty} 2^{-Nk} \left(\frac{1}{|2^k B(x, \rho(x))|^{1-\alpha s/n}} \int_{B_k(x, \rho(x))} |f(z)|^s dz \right)^{1/s}$$

where $B_k(x, \rho(x)) = 2C_0 \times 2^k B(x, \rho(x))$.

Let $\{Q_j\}_j$ be the family of critical balls as in Proposition 2.2. By (5), $C_0^{-1}\rho(x) \leq \rho(x_j) \leq C_0\rho(x)$ for all $x \in Q_j$. So, $|B(x, \rho(x))| \approx |Q_j|$ and $B_k(x, \rho(x)) \subset Q_j^k$ where

$Q_j^k = 2C_0 \times 2^k Q_j$. For $w^q \in A_{1+\frac{1}{(p/s)'}}$, using Hölder inequalities, we obtain

$$\begin{aligned}
\|G_{\alpha,s}f\|_{L^q(w^q)} &\lesssim \sum_{k=0}^{\infty} 2^{-Nk} \left(\sum_j \int_{Q_j} \left(\frac{1}{|2^k B(x, \rho(x))|^{1-\alpha s/n}} \int_{B_k(x, \rho(x))} |f(z)|^s dz \right)^{q/s} w^q(x) dx \right)^{1/q} \\
&\lesssim \sum_{k=0}^{\infty} 2^{-Nk} \left(\sum_j \int_{Q_j} \left(\frac{1}{|2^k Q_j|^{1-\alpha s/n}} \int_{Q_j^k} |f(z)|^s dz \right)^{q/s} w^q(x) dx \right)^{1/q} \\
&\lesssim \sum_{k=0}^{\infty} 2^{-Nk} \left(\sum_j \frac{w^q(Q_j)}{|2^k Q_j|^{q/s-\alpha q/n}} \left(\int_{Q_j^k} |f(z)|^s dz \right)^{q/s} \right)^{1/q} \\
&\lesssim \sum_{k=0}^{\infty} 2^{-Nk} \left(\sum_j \frac{1}{|2^k Q_j|^{q/s-\alpha q/n}} \right. \\
&\quad \times w^q(Q_j^k) \left(\int_{Q_j^k} (w^q)^{-\frac{s}{q}(\frac{p}{s})'} \right)^{\frac{q/s}{(p/s)'}} \left(\int_{Q_j^k} |f(z)|^p w^p(z) dz \right)^{q/p} \Big)^{1/q}.
\end{aligned} \tag{7}$$

Since $w^q \in A_{1+\frac{q/s}{(p/s)'}}^L$, we have, by (5),

$$w^q(Q_j^k) \left(\int_{Q_j^k} (w^q)^{-\frac{s}{q}(\frac{p}{s})'} \right)^{\frac{q/s}{(p/s)'}} \leq C |Q_j^k|^{1/s-\alpha/n} 2^{kN_0\theta \times (1/s-\alpha/n)}$$

for some $\theta > 0$.

This in combination with (7) gives

$$\begin{aligned}
\|G_{\alpha,s}f\|_{L^q(w^q)} &\lesssim \sum_k 2^{-Nk} \left(\sum_j \int_{Q_j^k} |f(z)|^p w^p(z) dz \right)^{1/p} \\
&\lesssim \sum_k 2^{-Nk} \left(\int_{\mathbb{R}^n} |f(z)|^p w^p(z) dz \right)^{1/p} \\
&\lesssim C \|f\|_{L^p(w^p)}
\end{aligned}$$

where in the last inequality we used (ii) in Proposition 2.2.

(ii) The proof of (ii) is similar to one of (i) and hence we omit details here.

For $0 \leq \alpha < n$, let M_α be the fractional maximal function defined by

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/n}} \int_B |f(y)| dy.$$

For $s \geq 1$, we define

$$M_{\alpha,s}f = \sup_{B \ni x} \left(\frac{1}{|B|^{1-s\alpha/n}} \int_B |f(y)|^s dy \right)^{1/s}$$

For a family of balls $\{Q_k\}_k$ given by Proposition 2.2, we define the operator $\widetilde{M}_{\alpha,s}$ as follows

$$\widetilde{M}_{\alpha,s}f = \sum_k \chi_{Q_k} M_{\alpha,s}(f \chi_{\widetilde{Q}_k}) \quad (8)$$

where $\widetilde{Q}_j = 4(2C_0^2 + 1)\gamma Q_j$ and γ is a constant in Proposition 2.3.

Remark 2.5 (i) For $s < p < \infty$ and $1/p - 1/q = \alpha/n$, it was proved in [MW] that $M_{\alpha,s}$ is bounded from $L^p(w^p)$ to $L^q(w^q)$ with $w^q \in A_{1+\frac{(q/s)}{(p/s)'}}$. This together with [BHS2, Proposition 4] shows that $\widetilde{M}_{\alpha,s}$ is bounded from $L^p(w^p)$ to $L^q(w^q)$ with $w^q \in A_{1+\frac{(q/s)}{(p/s)'}}^L$ here $s < p < \infty$ and $1/p - 1/q = \alpha/n$.

(ii) When $\alpha = 0$, we write \widetilde{M}_s instead of $\widetilde{M}_{0,s}$. The similar argument as in (i) also shows that for $p > s$ and $w \in A_{p/s}^L$, \widetilde{M}_s is bounded on $L^p(w)$.

3 Weighted estimates for commutators of singular integrals

3.1 Riesz transforms

3.1.1 Kernel estimates of Riesz transforms

In the sequel, let us remind that for the number N , we shall mean that N is a sufficiently large number and different from line to line.

Let K and K^* be the vector valued kernels of R and R^* respectively. The following propositions give some estimates on the kernels of R and R^* , see for example [Sh, GLP].

Proposition 3.1 a) If $V \in RH_q$ with $q > n/2$ then we have

(i) For every N there exists a constant C such that

$$|K^*(x, y)| \leq C \frac{(1 + \frac{|x-z|}{\rho(x)})^{-N}}{|x-z|^{n-1}} \left(\int_{B(z, |x-z|/4)} \frac{V(u)}{|u-z|^{n-1}} + \frac{1}{|x-z|} \right). \quad (9)$$

(ii) For every N and $0 < \delta < \min\{1, 2 - n/q\}$ there exists a constant C such that

$$|K^*(x, z) - K^*(y, z)| \leq C \frac{|x-y|^\delta (1 + \frac{|x-z|}{\rho(x)})^{-N}}{|x-z|^{n-1+\delta}} \left(\int_{B(z, |x-z|/4)} \frac{V(u)}{|u-z|^{n-1}} + \frac{1}{|x-z|} \right) \quad (10)$$

whenever $|x-y| < \frac{2}{3}|x-z|$.

b) If $V \in RH_q$ with $q \geq n$ then we have

(i) For every N there exists a constant C such that

$$|K(x, y)| \leq C \frac{(1 + \frac{|x-y|}{\rho(x)})^{-N}}{|x-y|^n}. \quad (11)$$

(ii) For every N and $0 < \delta < \min\{1, 1 - d/q\}$ there exists a constant C such that

$$|K^*(x, z) - K^*(y, z)| \leq C \frac{|x-y|^\delta (1 + \frac{|x-z|}{\rho(x)})^{-N}}{|x-z|^{n+\delta}} \quad (12)$$

whenever $|x-y| < \frac{2}{3}|x-z|$.

3.1.2 Commutators of Riesz transforms

The main result concerning the weighted estimates for R_b^* and R_b is formulated by the following theorem.

Theorem 3.2 *Let $b \in BMO_L^\theta$ with $\theta > 0$ and $V \in RH_q$.*

(i) *If $n/2 < q < n$ and s is such that $1/s = 1/q - 1/n$, the commutator R_b^* is bounded on $L^p(w)$ for $s' < p < \infty$ and $w \in A_{p/s'}^L$ and hence by duality R_b is bounded on $L^p(w)$ for $1 < p < s$ with $w^{-\frac{1}{p-1}} \in A_{p'/s'}^L$.*

(ii) *If $q \geq n$, the commutators R_b^* and R_b are bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p^L$.*

Proof: The proof of part (ii) is completely analogous to that of (i). Hence, we only provide the proof for (i) here and leave the second part to the interested readers.

(i) To prove (i), we exploit the strategy in [BHS1]. For any $s' < p < \infty$ and $w \in A_{p/s'}^L$, we have by Lemma 2.3

$$\begin{aligned} \|R_b^* f\|_{L^p(w)}^p &\leq \int_{\mathbb{R}^n} |M_{\rho, \beta}(R_b^* f)(x)|^p w(x) dx \\ &\lesssim \int_{\mathbb{R}^n} |M_{\rho, \gamma}^\#(R_b^* f)(x)|^p w(x) dx + \sum_k w(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |R_b^* f| \right)^p \end{aligned}$$

where $\{Q_k\}$ is a family of critical balls given in Lemma 2.3.

1. Estimate $\sum_k w(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |R_b^* f| \right)^p$

Let $s' < p_0 < p$ and $Q = B(x_0, \rho(x_0))$. Then we write

$$R_b^* f = (b - b_Q) R^* f - R^*((b - b_Q)f).$$

So, we have

$$\frac{1}{|2Q|} \int_{2Q} |R_b^* f| dx \leq \frac{1}{|2Q|} \int_{2Q} |(b - b_Q) R^* f| dx + \frac{1}{|2Q|} \int_{2Q} |R^*((b - b_Q)f)| dx := I_1 + I_2.$$

Let us estimate I_1 first. By Hölder inequality, we can write

$$\begin{aligned} I_1 &\lesssim \|b\|_\theta \left(\frac{1}{|2Q|} \int_{2Q} |R^* f|^{p_0} \right)^{1/p_0} \\ &\lesssim \|b\|_\theta \left(\left(\frac{1}{|2Q|} \int_{2Q} |R^* f_1|^{p_0} \right)^{1/p_0} + \left(\frac{1}{|2Q|} \int_{2Q} |R^* f_2|^{p_0} \right)^{1/p_0} \right) \\ &:= I_{11} + I_{12} \end{aligned}$$

where $f = f_1 + f_2$ with $f_1 = f \chi_{4Q}$.

Due to L^{p_0} -boundedness of R^* , one has

$$I_{11} \lesssim \left(\frac{1}{|2Q|} \int_{4Q} |f|^{p_0} \right)^{1/p_0} \lesssim \inf_{z \in Q} H_{p_0} f(z).$$

To estimate I_{12} , for $x \in 2Q$, due to (9), we have

$$\begin{aligned} R^* f_2(x) &\leq \int_{\mathbb{R}^n \setminus 4Q} |K^*(x, y) f(y)| dy \\ &\lesssim \int_{\mathbb{R}^n \setminus 4Q} \frac{(1 + \frac{|x-y|}{\rho(x)})^{-N}}{|x-y|^n} |f(y)| dy \\ &\quad + \int_{\mathbb{R}^n \setminus 4Q} \frac{(1 + \frac{|x-y|}{\rho(x)})^{-N}}{|x-y|^{n-1}} \int_{B(y, |x-y|/4)} \frac{V(u)}{|u-y|^{n-1}} |f(y)| du dy \\ &:= A_1(x) + A_2(x). \end{aligned}$$

To take care A_1 , note that $\rho(x) \approx \rho(x_0)$ and $|x-y| \approx |x_0-y|$ for all $x \in 2Q$ and $y \in \mathbb{R}^n \setminus 4Q$. So, decomposing $\mathbb{R}^n \setminus 4Q$ into annuli $2^{k+1} \setminus 2^k Q$, we have

$$\begin{aligned} A_1(x) &\lesssim \sum_{k \geq 2} \frac{2^{-kN}}{|2^k Q|} \int_{2^k Q} |f(y)| dy \\ &\lesssim \sum_{k \geq 2} 2^{-kN} \left(\frac{1}{|2^k Q|} \int_{2^k Q} |f(y)|^{p_0} dy \right)^{1/p_0} \\ &\lesssim \inf_{z \in Q} H_{p_0} f(z) \end{aligned}$$

for all $x \in 2Q$.

For the term A_2 , by decomposing $\mathbb{R}^n \setminus 4Q$ into annuli $2^{k+1} \setminus 2^k Q$, we get that

$$\begin{aligned} A_2(x) &\lesssim \sum_{k \geq 2} \frac{2^{-kN}}{|2^k Q|^{1-1/n}} \int_{2^k Q} |f(y)| \int_{2^{k+2} Q} \frac{V(u)}{|u-y|^{n-1}} du dy \\ &\lesssim \sum_{k \geq 2} \frac{2^{-kN}}{|2^k Q|^{1-1/n}} \int_{2^k Q} |f(y)| \mathcal{I}_1(V \chi_{2^{k+2} Q})(y) dy \end{aligned}$$

where $\mathcal{I}_1 = (-\Delta)^{-1/2}$.

Let us remind that \mathcal{I}_1 is $L^{q_0} - L^{p'_0}$ boundedness with $1/p'_0 = 1/q_0 - 1/n$. This together with Hölder inequality gives

$$\begin{aligned} A_2(x) &\lesssim \sum_{k \geq 2} \frac{2^{-kN}}{|2^k Q|^{1-1/n}} \int_{\mathbb{R}^n} |(f \chi_{2^k Q})(y)| \mathcal{I}_1(V \chi_{2^{k+2} Q})(y) dy \\ &\lesssim \sum_{k \geq 2} \frac{2^{-kN}}{|2^k Q|^{1-1/n}} \left(\int_{2^k Q} |f|^{p_0} \right)^{1/p_0} \|\mathcal{I}_1(V \chi_{2^{k+2} Q})\|_{L^{p'_0}} \\ &\lesssim \sum_{k \geq 2} \frac{2^{-kN}}{|2^k Q|^{1-1/n-1/p_0}} \left(\frac{1}{|2^k Q|} \int_{2^k Q} |f|^{p_0} \right)^{1/p_0} \|V \chi_{2^{k+2} Q}\|_{L^{q_0}} \end{aligned}$$

Noting that if we choose p_0 to be close enough to s' then $V \in RH_{q_0}$. This in combination with the fact that V is a doubling measure gives

$$\begin{aligned} \|V \chi_{2^{k+2} Q}\|_{L^{q_0}} &\lesssim \frac{1}{|2^k Q|^{1/q'_0}} \int_{2^{k+2} Q} V \\ &\lesssim \frac{2^{k\kappa}}{|2^k Q|^{1/q'_0}} \int_Q V \quad \text{for some } \kappa > 0 \\ &\lesssim \frac{2^{k\kappa}}{|2^k Q|^{1/q'_0-1+2/n}}. \end{aligned}$$

Hence,

$$\begin{aligned} A_2(x) &\lesssim \sum_{k \geq 2} \frac{2^{k\kappa}}{|2^k Q|^{1/q'_0-1+2/n}} \frac{2^{-kN}}{|2^k Q|^{1-1/n-1/p_0}} \left(\frac{1}{|2^k Q|} \int_{2^k Q} |f|^{p_0} \right)^{1/p_0} \\ &\lesssim \sum_{k \geq 2} 2^{-k(N-\kappa)} \left(\frac{1}{|2^k Q|} \int_{2^k Q} |f|^{p_0} \right)^{1/p_0} \\ &\lesssim \inf_{z \in Q} H_{p_0} f(z) \end{aligned}$$

for all $x \in 2Q$.

From the estimates of A_1 and A_2 , we obtain $I_{12} \lesssim \inf_{z \in Q} H_{p_0} f(z)$.

The term I_2 can be estimated in the same line with I_1 . Using the decomposition $f = f_1 + f_2$ again, one gets that

$$I_2 \leq \left(\frac{1}{|2Q|} \int_{2Q} |R^*((b - b_Q)f_1)(y)| dy \right) + \left(\frac{1}{|2Q|} \int_{2Q} |R^*((b - b_Q)f_2)(y)| dy \right) := I_{21} + I_{22}.$$

Choose $s' < r < p_0$. Using Hölder inequality and L^r -boundedness of R^* , we have

$$\begin{aligned}
I_{21} &\lesssim \left(\frac{1}{|2Q|} \int_{2Q} |R^*((b - b_Q)f_1)(y)|^r dy \right)^{1/r} \\
&\lesssim \left(\frac{1}{|2Q|} \int_{4Q} |((b - b_Q)f_1)(y)|^r dy \right)^{1/r} \\
&\lesssim \left(\frac{1}{|2Q|} \int_{4Q} |f_1(y)|^{p_0} dy \right)^{1/p_0} \left(\frac{1}{|2Q|} \int_{4Q} (b - b_Q)^\nu dy \right)^{1/\nu} \text{ for some } \nu > r \\
&\lesssim \|b\|_\theta \inf_{z \in Q} H_{p_0} f(z).
\end{aligned}$$

The estimate of $I_{22} \lesssim \|b\|_\theta \inf_{z \in Q} H_{p_0} f(z)$ can be taken care similarly to ones of I_{12} and I_{21} . So we omit the details here. To sum up, it had proved that for any critical ball Q , we have

$$\frac{1}{|2Q|} \int_{2Q} |R_b^* f| dx \lesssim \|b\|_\theta \inf_{z \in Q} H_{p_0} f(z). \quad (13)$$

Return to the estimate of $\sum_k w(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |R_b^* f| \right)^p$. Due to (19), we have

$$\begin{aligned}
&\sum_k w(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |R_b^* f| \right)^p \\
&\lesssim \sum_k \|b\|_\theta^p w(Q_k) \left(\inf_{z \in Q} H_{p_0} f(z) \right)^p \\
&\lesssim \sum_k \|b\|_\theta^p \int_{Q_k} |H_{p_0} f(z)|^p w(z) dz \\
&\lesssim \|b\|_\theta^p \int_{\mathbb{R}^n} |H_{p_0} f(z)|^p w(z) dz \\
&\lesssim \|b\|_\theta^p \|f\|_{L^p(w)}^p \text{ due to Proposition 2.4}
\end{aligned}$$

for all $w \in A_{p/p_0}^L$. Letting $p_0 \rightarrow s'$, we obtain

$$\sum_k w(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |R_b^* f| \right)^p \lesssim \|b\|_\theta^p \|f\|_{L^p(w)}^p$$

for all $w \in A_{p/s'}^L$.

2. Estimate $\int_{\mathbb{R}^n} |M_{\rho, \gamma}^\sharp(R_b^* f)(x)|^p w(x) dx$

For any ball $B(x_0, r)$ with $r \leq \gamma\rho(x_0)$ and $x \in B$, we write

$$\begin{aligned} & \frac{1}{|B|} \int_B |R_b^* f(x) - (R_b^* f)_B| dx \\ & \leq \frac{2}{|B|} \int_B |(b - b_B) R^* f(x)| dx + \frac{2}{|B|} \int_B |R^*((b - b_B)f_1)(x)| dx \\ & \quad + \frac{1}{|B|} \int_B |R^*((b - b_B)f_2)(x) - (R^*((b - b_B)f_2))_B| dx \\ & := E_1 + E_2 + E_3. \end{aligned}$$

where $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$.

Let $s' < p_0 < p$, Hölder inequality and Proposition 1.3 show that

$$\begin{aligned} E_1 & \lesssim \left(\frac{1}{|B|} \int_B |b - b_B|^{p'_0} \right)^{1/p'_0} \left(\frac{1}{|B|} \int_B |R^* f|^{p_0} \right)^{1/p_0} \\ & \lesssim \|b\|_\theta \left(\frac{1}{|B|} \int_B |R^* f|^{p_0} \right)^{1/p_0}. \end{aligned}$$

For any critical ball Q_j such that $x \in Q_j \cap B$. It can be verified that $B \subset \widetilde{Q}_j$. This yields that

$$E_1 \lesssim \|b\|_\theta \times \inf_{y \in B} \widetilde{M}_{p_0}(R^* f)(y).$$

For some $s' < r < p_0 < p$, Hölder inequality and Proposition 1.3 again tell us that

$$\begin{aligned} E_2 & \lesssim \left(\frac{1}{|B|} \int_B |R^*((b - b_B)f_1)|^r \right)^{1/r} \\ & \lesssim \left(\frac{1}{|B|} \int_{2B} |(b - b_B)f_1|^r \right)^{1/r} \\ & \lesssim \left(\frac{1}{|B|} \int_{2B} |(b - b_B)|^\nu \right)^{1/\nu} \left(\frac{1}{|B|} \int_{2B} |f|^{p_0} \right)^{1/p_0} \text{ for some } \nu > r \\ & \lesssim \|b\|_\theta \times \inf_{y \in B} \widetilde{M}_{p_0}(f)(y). \end{aligned}$$

To estimate E_3 , we need to show that

$$\int_{\mathbb{R}^n \setminus 2B} |K^*(x, z) - K^*(y, z)| |b(z) - b_B| |f(z)| dz \lesssim \|b\|_\theta \left(\inf_{u \in B} H_{\gamma, p_0} f(u) + \inf_{u \in B} \widetilde{M}_{p_0}(f)(u) \right) \quad (14)$$

for all f and $x, y \in B$. If this holds, then we have

$$\begin{aligned} E_3 & \lesssim \frac{1}{|B|^2} \int_B \int_B \left(\int_{\mathbb{R}^n \setminus 2B} |K^*(u, z) - K^*(y, z)| |b(z) - b_B| |f(z)| dz \right) dy du \\ & \lesssim \|b\|_\theta (H_{\gamma, p_0} f(x) + \widetilde{M}_{p_0}(f)(x)). \end{aligned}$$

These three estimates of E_1, E_2 and E_3 give

$$M_{\rho, \gamma}^{\sharp}(R_b^*)(x) \lesssim \|b\|_{\theta}(\widetilde{M}_{p_0}(R^*f)(x) + H_{\gamma, p_0}(x) + \widetilde{M}_{p_0}(f)(x)).$$

This implies

$$\|M_{\rho, \gamma}^{\sharp}(R_b^*)\|_{L^p(w)} \lesssim \|b\|_{\theta}(\|\widetilde{M}_{p_0}(R^*f)\|_{L^p(w)} + \|H_{\gamma, p_0}f\|_{L^p(w)} + \|\widetilde{M}_{p_0}(f)\|_{L^p(w)}).$$

Since $\widetilde{M}_{p_0}, H_{\gamma, p_0}$ and R^*f is bounded on $L^p(w)$ for all $w \in A_{p/p_0}^L$. Letting $p_0 \rightarrow s'$, we obtain the desired results.

Proof of (14): We adapt some ideas of [BHS1, Lemma 6] to our present situation. Setting $Q = B(x_0, \gamma\rho(x_0))$, due to (10) and the fact that $\rho(x) \approx \rho(x_0)$ and $|x-z| \approx |x_0-z|$, we get

$$\int_{\mathbb{R}^n \setminus 2B} |K^*(x, z) - K^*(y, z)| |b(z) - b_B| |f(z)| dz \lesssim K_1 + K_2 + K_3 + K_4$$

where

$$\begin{aligned} K_1 &= r^{\delta} \int_{Q \setminus 2B} \frac{|f(z)(b(z) - b_B)|}{|x_0 - z|^{n+\delta}} dz, \\ K_2 &= r^{\delta} \rho(x_0)^N \int_{Q^c} \frac{|f(z)(b(z) - b_B)|}{|x_0 - z|^{n+\delta+N}} dz, \\ K_3 &= r^{\delta} \int_{Q \setminus 2B} \frac{|f(z)(b(z) - b_B)|}{|x_0 - z|^{n-1+\delta}} \int_{B(x_0, 4|x_0-z|)} \frac{V(u)}{|u - z|^{n-1}} du dz, \end{aligned}$$

and

$$K_4 = r^{\delta} \rho(x_0)^N \int_{Q^c} \frac{|f(z)(b(z) - b_B)|}{|x_0 - z|^{n-1+\delta+N}} \int_{B(x_0, 4|x_0-z|)} \frac{V(u)}{|u - z|^{n-1}} du dz.$$

Let j_0 be the smallest integer so that $2^{j_0}r \geq \gamma\rho(x_0)$. Splitting $Q \setminus 2B$ into annuli $2^{k+1}B \setminus 2^k B$ for $k = 1, \dots, j_0$, we obtain

$$K_1 \lesssim \sum_{k=1}^{j_0} \frac{2^{-k\delta}}{|2^k B|} \int_{2^k B} |f(z)(b(z) - b_B)| dz.$$

By Hölder inequality and Proposition 1.3,

$$K_1 \lesssim \sum_{k=1}^{j_0} k 2^{-k\delta} \|b\|_{\theta} \left(\frac{1}{|2^k B|} \int_{2^k B} |f(z)|^{p_0} dz \right)^{1/p_0}$$

Remind that if $x \in B \cap Q_j$ then $2^k B \subset \widetilde{Q}_j$ where Q_j and \widetilde{Q}_j are balls in (8). Therefore,

$$K_1 \lesssim \sum_{k=1}^{j_0} k 2^{-k\delta} \|b\|_{\theta} \inf_{u \in B} \widetilde{M}_{p_0} f(u).$$

Splitting Q^c into annuli and then applying Hölder inequality and Proposition 1.3 again, we obtain

$$\begin{aligned} K_2 &\lesssim \left(\frac{\rho(x_0)}{2^{j_0}r}\right)^N \sum_{k=j_0-1}^{\infty} \frac{2^{-k\delta-(k-j_0)N}}{|2^{k-j_0+1}2^{j_0-1}B|} \int_{2^k B} |f(z)(b(z) - b_B)| dz \\ &\lesssim \|b\|_{\theta} \left(\frac{\rho(x_0)}{2^{j_0}r}\right)^N \sum_{k=j_0-1}^{\infty} k 2^{-k\delta-(k-j_0)(N-\theta')} \left(\frac{1}{|2^{k-j_0+1}2^{j_0-1}B|} \int_{2^k B} |f(z)|^{p_0} dz\right)^{1/p_0}. \end{aligned}$$

Since $2^{j_0}r \geq \gamma\rho(x_0) \geq 2^{j_0-1}r$, we get that

$$\begin{aligned} K_2 &\lesssim \|b\|_{\theta} \sum_{k=j_0-1}^{\infty} 2^{-(k-j_0)(N-\theta')} \left(\frac{1}{|2^{k-j_0+1}Q|} \int_{2^{k-j_0+1}Q} |f(z)|^{p_0} dz\right)^{1/p_0} \\ &\lesssim \|b\|_{\theta} \sum_{l=1}^{\infty} 2^{-l(N-\theta')} \left(\frac{1}{|2^l Q|} \int_{2^l Q} |f(z)|^{p_0} dz\right)^{1/p_0} \\ &\lesssim \|b\|_{\theta} \inf_{u \in B} H_{\gamma, p_0} f(u) \text{ since } Q = \gamma B(x_0, \rho(x_0)). \end{aligned}$$

It can be verified that

$$K_3 \lesssim \sum_{k=2}^{j_0} \frac{2^{-k\delta}}{|2^k B|^{1-1/n}} \int_{2^k B} |f(z)(b(z) - b_B)| \mathcal{I}_1(V\chi_{2^{k+2}B})(z) dz$$

Let $s' < q_0 < p_0$, $\nu = \frac{p_0 q_0}{p_0 - q_0}$ and r such that $1/r = 1/q'_0 + 1/n$ then by Hölder inequality and Proposition 1.3

$$\begin{aligned} K_3 &\lesssim \|b\|_{\theta} \sum_{k=2}^{j_0} \frac{k 2^{-k\delta}}{|2^k B|^{-1/n+1/q'_0}} \left(\frac{1}{|2^k B|} \int_{2^k B} |f|^{p_0}\right)^{1/p_0} \|\mathcal{I}_1(V\chi_{2^{k+2}B})\|_{L^{q'_0}} \\ &\lesssim \|b\|_{\theta} \sum_{k=2}^{j_0} \frac{k 2^{-k\delta}}{|2^k B|^{-1/n+1/q'_0}} \left(\frac{1}{|2^k B|} \int_{2^k B} |f|^{p_0}\right)^{1/p_0} \|V\chi_{2^{k+2}B}\|_{L^r} \\ &\lesssim \|b\|_{\theta} \sum_{k=2}^{j_0} \frac{k 2^{-k\delta}}{|2^k B|^{-1/n+1/q'_0}} \|V\chi_{2^{k+2}B}\|_{L^r} \inf_{u \in B} \widetilde{M}_{p_0} f(u) \end{aligned}$$

Noting that we can choose q_0 so that $V \in RH_r$, then we have

$$\begin{aligned} \|V\chi_{2^{k+2}B}\|_{L^r} &\lesssim \left(\int_Q V(z)^r dz\right)^{1/r} \lesssim \frac{1}{|Q|^{1-1/r}} \int_Q V(z) dz \\ &\lesssim \frac{1}{|Q|^{2/n-1/r}} \lesssim \frac{1}{|2^k B|^{2/n-1/r}} \end{aligned}$$

for all $k = 2, \dots, j_0$.

So,

$$K_3 \lesssim \|b\|_\theta \sum_{k=2}^{j_0} k 2^{-k\delta} \widetilde{M}_{p_0} f(x) := C \|b\|_\theta \inf_{u \in B} \widetilde{M}_{p_0} f(u).$$

The similar arguments to ones used to obtain the estimates K_2 and K_3 gives

$$K_4 \lesssim \|b\|_\theta \inf_{u \in B} H_{\gamma, p_0} f(u).$$

This completes our proof.

3.2 Fractional integrals

3.2.1 Kernel estimates of fractional integrals

Let K_α be the kernel of I_α . The following results give the estimates on the kernel $K_\alpha(x, y)$.

Proposition 3.3 *If $V \in RH_q$ with $q > n/2$ then we have*

(i) *For every N there exists a constant C such that*

$$|K_\alpha(x, y)| \leq C \frac{(1 + \frac{|x-y|}{\rho(x)})^{-N}}{|x-y|^{n-\alpha}}. \quad (15)$$

(ii) *There is a number $\delta > 0$ such that for every N there exists a constant C such that*

$$|K_\alpha(x, y) - K_\alpha(x, z)| \leq C \frac{|y-z|^\delta (1 + \frac{|x-y|}{\rho(x)})^{-N}}{|x-y|^{n+\delta-\alpha}} \quad (16)$$

whenever $|y-z| < \frac{1}{4}|x-y|$.

To prove Proposition 3.3, we need the following estimates of the heat kernels of e^{-tL} , see [DZ2, p.12]

Proposition 3.4 *Let $p_t(x, y)$ be the kernels associated to the semigroups $\{e^{-tL}\}_{t>0}$. If $V \in RH_q$ with $q > n/2$ then we have*

(i) *For every $N > 0$ there exists a constant C such that*

$$|p_t(x, y)| \leq \frac{C}{t^{n/2}} \exp\left(-c \frac{|x-y|^2}{t}\right) \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}; \quad (17)$$

(ii) *There is a number $\delta > 0$ such that for every N there exists a constant C such that*

$$\begin{aligned} & |p_t(x, y) - p_t(x, z)| + |p_t(y, x) - p_t(z, x)| \\ & \leq \frac{C}{t^{n/2}} \left(\frac{|y-z|}{\sqrt{t}}\right)^\delta \exp\left(-c \frac{|x-y|^2}{t}\right) \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \end{aligned} \quad (18)$$

whenever $|y-z| < \frac{1}{2}|x-y|$.

Proof of Proposition 3.3: (i) We have, by (17),

$$\begin{aligned}
|K_\alpha(x, y)| &\leq \int_0^\infty |t^{\alpha/2} p_t(x, y)| \frac{dt}{t} \\
&\lesssim \int_0^{|x-y|^2} \frac{t^{\alpha/2}}{t^{n/2}} \exp\left(-c \frac{|x-y|^2}{t}\right) \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \frac{dt}{t} \\
&\quad + \int_{|x-y|^2}^\infty \frac{t^{\alpha/2}}{t^{n/2}} \exp\left(-c \frac{|x-y|^2}{t}\right) \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \frac{dt}{t} \\
&= I_1 + I_2.
\end{aligned}$$

Let us estimate I_2 first. Since $t > |x-y|^2$, we have, for $\epsilon > 0$ so that $n > \alpha + \epsilon$,

$$\begin{aligned}
I_2 &\lesssim \int_{|x-y|^2}^\infty \frac{t^{\alpha/2}}{t^{n/2}} \exp\left(-c \frac{|x-y|^2}{t}\right) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} \frac{dt}{t} \\
&\lesssim \int_{|x-y|^2}^\infty \frac{t^{\alpha/2}}{t^{n/2}} \left(\frac{t}{|x-y|^2}\right)^{n/2-\alpha/2-\epsilon} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} \frac{dt}{t} \\
&\lesssim \frac{1}{|x-y|^{n-\alpha}} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}.
\end{aligned}$$

For I_1 , we have

$$\begin{aligned}
I_1 &\lesssim \int_0^{|x-y|^2} \frac{t^{\alpha/2}}{t^{n/2}} \left(\frac{t}{|x-y|^2}\right)^{n/2+N/2} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \frac{dt}{t} \\
&\lesssim \int_0^{|x-y|^2} \frac{t^{\alpha/2}}{|x-y|^n} \left(\frac{\sqrt{t}}{|x-y|}\right)^N \left(\frac{\sqrt{t} + \rho(x)}{\rho(x)}\right)^{-N} \frac{dt}{t} \\
&\lesssim \int_0^{|x-y|^2} \frac{t^{\alpha/2}}{|x-y|^n} \left(\frac{\sqrt{t} + \rho(x)}{|x-y| + \rho(x)}\right)^N \left(\frac{\sqrt{t} + \rho(x)}{\rho(x)}\right)^{-N} \frac{dt}{t} \\
&\lesssim \frac{1}{|x-y|^{n-\alpha}} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}.
\end{aligned}$$

This completes (i).

(ii) For $|y-z| < \frac{1}{4}|x-y|$, using (18) gives

$$\begin{aligned}
|K_\alpha(x, y) - K_\alpha(x, z)| &\lesssim \int_0^\infty \frac{t^{\alpha/2}}{t^{n/2}} \left(\frac{|y-z|}{\sqrt{t}}\right)^\delta \exp\left(-c \frac{|x-y|^2}{t}\right) \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \frac{dt}{t} \\
&\lesssim \left(\frac{|y-z|}{|x-y|}\right)^\delta \int_0^\infty \frac{t^{\alpha/2}}{t^{n/2}} \exp\left(-c' \frac{|x-y|^2}{t}\right) \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \frac{dt}{t} \\
&\lesssim \left(\frac{|y-z|}{|x-y|}\right)^\delta \left(\int_0^{|x-y|^2} \dots + \int_{|x-y|^2}^\infty \dots\right) \\
&= \left(\frac{|y-z|}{|x-y|}\right)^\delta (II_1 + II_2).
\end{aligned}$$

At this stage, repeating the arguments in (i), we obtain (ii).

3.2.2 Commutators of fractional integrals

We are now in position to state the result concerning the weighted estimates for I_α^b .

Theorem 3.5 *Let $b \in BMO_L^\theta$ with $\theta > 0$ and $V \in RH_q$ with $q > n/2$. Then the commutator I_α^b is bounded from $L^p(w^p)$ to $L^q(w^q)$ for $1 < p < \infty$, $1/p - 1/q = \alpha/n$ and $w^q \in A_{1+\frac{q}{p}}^L$.*

Proof: The strategy of the proof for Theorem 3.5 is similar to that of Theorem 3.2. For any $1 < s < p < \infty$, $1/p - 1/q = \alpha/n$ and $w^q \in A_{1+\frac{(q/s)}{(p/s)'}}^L$, we have by Lemma 2.3

$$\begin{aligned} \|I_\alpha^b f\|_{L^q(w^q)}^q &\leq \int_{\mathbb{R}^n} |M_{\rho,\beta}(I_\alpha^b f)(x)|^q w^q(x) dx \\ &\lesssim \int_{\mathbb{R}^n} |M_{\rho,\gamma}^\sharp(I_\alpha^b f)(x)|^q w^q(x) dx + \sum_k w^q(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |I_\alpha^b f| \right)^q \end{aligned}$$

where $\{Q_k\}$ is a family of critical balls given in Proposition 2.3.

So, to obtain the weighted estimates for I_α^b , we need only to consider $\int_{\mathbb{R}^n} |M_{\rho,\gamma}^\sharp(I_\alpha^b f)(x)|^q w^q(x) dx$ and $\sum_k w^q(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |I_\alpha^b f| \right)^q$.

Step 1. Estimate $\sum_k w^q(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |I_\alpha^b f| \right)^q$

Let $1 < s < p$, $1/s - 1/v = \alpha/n$ and $Q = B(x_0, \rho(x_0))$. We have

$$I_\alpha^b f = (b - b_Q)I_\alpha f - I_\alpha((b - b_Q)f).$$

Hence,

$$\frac{1}{|2Q|} \int_{2Q} |I_\alpha^b f| dx \leq \frac{1}{|2Q|} \int_{2Q} |(b - b_Q)I_\alpha f| dx + \frac{1}{|2Q|} \int_{2Q} |I_\alpha((b - b_Q)f)| dx := I_1 + I_2.$$

To take care I_1 , using Hölder inequality and Proposition 1.3, we get that

$$\begin{aligned} I_1 &\lesssim \|b\|_\theta \left(\frac{1}{|2Q|} \int_{2Q} |I_\alpha f|^v \right)^{1/v} \\ &\lesssim \|b\|_\theta \left(\left(\frac{1}{|2Q|} \int_{2Q} |I_\alpha f_1|^v \right)^{1/v} + \left(\frac{1}{|2Q|} \int_{2Q} |I_\alpha f_2|^v \right)^{1/v} \right) \\ &:= I_{11} + I_{12} \end{aligned}$$

where $f = f_1 + f_2$ with $f_1 = f\chi_{4Q}$.

Since I_α is $L^s - L^v$ bounded, one has

$$I_{11} \lesssim \left(\frac{1}{|2Q|^{1-\alpha s/n}} \int_{4Q} |f|^s \right)^{1/s} \lesssim \inf_{z \in Q} G_{\alpha,s} f(z).$$

To estimate I_{12} , for $x \in 2Q$, (15) implies that

$$\begin{aligned} I_\alpha f_2(x) &\leq \int_{\mathbb{R}^n \setminus 4Q} |K_\alpha(x, y) f(y)| dy \\ &\lesssim \int_{\mathbb{R}^n \setminus 4Q} \frac{(1 + \frac{|x-y|}{\rho(x)})^{-N}}{|x-y|^{n-\alpha}} |f(y)| dy. \end{aligned}$$

In this situation, we have $\rho(x) \approx \rho(x_0)$ and $|x-y| \approx |x_0-y|$ for all $x \in 2Q$ and $y \in \mathbb{R}^n \setminus 4Q$. So, decomposing $\mathbb{R}^n \setminus 4Q$ into annuli $2^{k+1} \setminus 2^k Q$, we have, by Hölder inequality,

$$\begin{aligned} I_\alpha f_2(x) &\lesssim \sum_{k \geq 2} \frac{2^{-kN}}{|2^k Q|^{1-\alpha/n}} \int_{2^k Q} |f(y)| dy \\ &\lesssim \sum_{k \geq 2} 2^{-kN} \left(\frac{1}{|2^k Q|^{1-\alpha s/n}} \int_{2^k Q} |f(y)|^s dy \right)^{1/s} \\ &\lesssim \inf_{z \in Q} G_{\alpha, s} f(z) \end{aligned}$$

for all $x \in 2Q$. Hence $I_{12} \lesssim \inf_{z \in Q} G_{\alpha, s} f(z)$.

The estimate for I_2 can be proceeded in the same line with one of I_1 . The decomposition $f = f_1 + f_2$ gives

$$I_2 \leq \left(\frac{1}{|2Q|} \int_{2Q} |I_\alpha((b - b_Q) f_1)(y)| dy \right) + \left(\frac{1}{|2Q|} \int_{2Q} |I_\alpha((b - b_Q) f_2)(y)| dy \right) := I_{21} + I_{22}.$$

Choose $1 < r < s < p$ and $1/r - 1/r_0 = \alpha/n$. Using Hölder inequality, Proposition 1.3 and $L^r - L^{r_0}$ -boundedness of I_α , we have

$$\begin{aligned} I_{21} &\lesssim \left(\frac{1}{|2Q|} \int_{2Q} |I_\alpha((b - b_Q) f_1)(y)|^{r_0} dy \right)^{1/r_0} \\ &\lesssim \frac{1}{|2Q|^{-\alpha/n}} \left(\frac{1}{|2Q|} \int_{4Q} |((b - b_Q) f_1)(y)|^r dy \right)^{1/r} \\ &\lesssim \frac{1}{|2Q|^{-\alpha/n}} \left(\frac{1}{|2Q|} \int_{4Q} |f_1(y)|^s dy \right)^{1/s} \left(\frac{1}{|2Q|} \int_{4Q} (b - b_Q)^\nu dy \right)^{1/\nu} \text{ for some } \nu > r \\ &\lesssim \|b\|_\theta \inf_{z \in Q} G_{\alpha, s} f(z). \end{aligned}$$

The estimate $I_{22} \lesssim \|b\|_\theta \inf_{z \in Q} G_{\alpha, s} f(z)$ can be obtained by the similar approach to ones of I_{12} and I_{21} . So we omit the details here.

To sum up, for any critical ball Q , we have

$$\frac{1}{|2Q|} \int_{2Q} |I_\alpha^b f| dx \lesssim \|b\|_\theta \inf_{z \in Q} G_{\alpha, s} f(z). \quad (19)$$

Return to the estimate of $\sum_k w^q(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |I_\alpha^b| \right)^q$. Due to (19), we have

$$\begin{aligned}
\sum_k w^q(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |I_\alpha^b f| \right)^q &\lesssim \sum_k \|b\|_\theta^q w^q(Q_k) \left(\inf_{z \in Q} G_{\alpha,s} f(z) \right)^q \\
&\lesssim \sum_k \|b\|_\theta^q \int_{Q_k} |G_{\alpha,s} f(z)|^q w^q(z) dz \\
&\lesssim \|b\|_\theta^q \int_{\mathbb{R}^n} |G_{\alpha,s} f(z)|^q w^q(z) dz \\
&\lesssim \|b\|_\theta^q \|f\|_{L^p(w^p)}^q \text{ due to Proposition 2.4}
\end{aligned}$$

for all $w^q \in A_{1+\frac{q/s}{(p/s)'}}^L$. Letting $s \rightarrow 1$, we obtain

$$\sum_k w^q(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |I_\alpha^b| \right)^q \lesssim \|b\|_\theta^p \|f\|_{L^p(w^p)}^q$$

for all $w^q \in A_{1+\frac{q}{p'}}^L$.

Step 2. Estimate $\int_{\mathbb{R}^n} |M_{\rho,\gamma}^\sharp(I_\alpha^b f)(x)|^q w^q(x) dx$

For any ball $B(x_0, r)$ with $r \leq \gamma\rho(x_0)$ and $x \in B$, we write

$$\begin{aligned}
&\frac{1}{|B|} \int_B |I_\alpha^b f(x) - (I_\alpha^b f)_B| dx \\
&\leq \frac{2}{|B|} \int_B |(b - b_B) I_\alpha f(x)| dx + \frac{2}{|B|} \int_B |I_\alpha((b - b_B) f_1)(x)| dx \\
&\quad + \frac{1}{|B|} \int_B |I_\alpha((b - b_B) f_2)(x) - (I_\alpha((b - b_B) f_2))_B| dx \\
&:= E_1 + E_2 + E_3.
\end{aligned}$$

where $f = f_1 + f_2$ with $f_1 = f \chi_{2B}$.

Applying Hölder inequality and Proposition 1.3, we have

$$\begin{aligned}
E_1 &\lesssim \left(\frac{1}{|B|} \int_B |b - b_B|^{s'} \right)^{1/s'} \left(\frac{1}{|B|} \int_B |I_\alpha f|^s \right)^{1/s} \\
&\lesssim \|b\|_\theta \left(\frac{1}{|B|} \int_B |I_\alpha f|^s \right)^{1/s}.
\end{aligned}$$

For any critical ball Q_j such that $x \in Q_j \cap B$, it is easy to see that $B \subset \tilde{Q}_j$. Therefore,

$$E_1 \lesssim \|b\|_\theta \times \inf_{u \in B} \widetilde{M}_s(I_\alpha f)(u).$$

For some $1 < r < s < p$ and $1/r - 1/r_0 = \alpha/n$, Hölder inequality and Proposition 1.3 again tell us that

$$\begin{aligned}
E_2 &\lesssim \left(\frac{1}{|B|} \int_B |I_\alpha((b - b_B)f_1)|^{r_0} \right)^{1/r_0} \\
&\lesssim \frac{1}{|B|^{-\alpha/n}} \left(\frac{1}{|B|} \int_{2B} |(b - b_B)f_1|^r \right)^{1/r} \\
&\lesssim \frac{1}{|B|^{-\alpha/n}} \left(\frac{1}{|B|} \int_{2B} |(b - b_B)|^\nu \right)^{1/\nu} \left(\frac{1}{|B|} \int_{2B} |f|^s \right)^{1/s} \text{ for some } \nu > r \\
&\lesssim \|b\|_\theta \times \inf_{u \in B} \widetilde{M}_{\alpha,s}(f)(u).
\end{aligned}$$

Before taking care E_3 , we need the to show that

$$\int_{\mathbb{R}^n \setminus 2B} |K_\alpha(x, z) - K_\alpha(y, z)| |b(z) - b_B| |f(z)| dz \lesssim \|b\|_\theta \left(\inf_{u \in B} G_{\gamma,\alpha,s} f(u) + \inf_{u \in B} \widetilde{M}_{\alpha,s}(f)(u) \right) \quad (20)$$

for all f and $x, y \in B$.

If this holds, then we have

$$\begin{aligned}
E_3 &\lesssim \frac{1}{|B|^2} \int_B \int_B \left(\int_{\mathbb{R}^n \setminus 2B} |K_\alpha(u, z) - K_\alpha(y, z)| |b(z) - b_B| |f(z)| dz \right) dy du \\
&\lesssim \|b\|_\theta (G_{\gamma,\alpha,s}(x) + \widetilde{M}_{\alpha,s}(f)(x)).
\end{aligned}$$

These three estimates of E_1, E_2 and E_3 give that

$$M_{\rho,\gamma}^\sharp(I_\alpha^b)(x) \lesssim \|b\|_\theta (\widetilde{M}_{\alpha,s}(f)(x) + G_{\gamma,\alpha,s} f(x) + \widetilde{M}_s(I_\alpha f)(x)).$$

This implies

$$\|M_{\rho,\gamma}^\sharp(I_\alpha^b)\|_{L^q(w^q)} \lesssim \|b\|_\theta (\|\widetilde{M}_{\alpha,s}(f)\|_{L^q(w^q)} + \|G_{\gamma,\alpha,s} f\|_{L^q(w^q)} + \|\widetilde{M}_s(I_\alpha f)\|_{L^q(w^q)}).$$

Since $\widetilde{M}_{\alpha,s}$ and $G_{\gamma,\alpha,s}$ are bounded form $L^p(w^p)$ to $L^q(w^q)$ for all $1 < p < n/\alpha$, $1/p - 1/q = \alpha/n$ and $w^q \in A_{1+(q/s)/(p/s)}^L$, we have

$$\|\widetilde{M}_{\alpha,s}(f)\|_{L^q(w^q)} + \|G_{\gamma,\alpha,s} f\|_{L^q(w^q)} \lesssim \|f\|_{L^p(w^p)}.$$

For the last term $\|\widetilde{M}_s(I_\alpha f)\|_{L^q(w^q)}$, from the weighted estimates of I_α and \widetilde{M}_s (see Remark 2.5) and the fact that $A_{1+\frac{(q/s)}{(p/s)'}} \subset A_{q/s}$, one gets that

$$\|\widetilde{M}_s(I_\alpha f)\|_{L^q(w^q)} \lesssim \|I_\alpha f\|_{L^q(w^q)} \lesssim \|f\|_{L^p(w^p)}.$$

Hence,

$$\|M_{\rho,\gamma}^\sharp(I_\alpha^b)\|_{L^q(w^q)} \lesssim \|b\|_\theta \|f\|_{L^p(w^p)}$$

for all $w^q \in A_{1+\frac{(q/s)}{(p/s)'}}$ and $1 < s < p$.

Letting $s \rightarrow 1$, we obtain the desired results.

Proof of (20): Setting $Q = B(x_0, \gamma\rho(x_0))$, due to (16) and the fact that $\rho(x) \approx \rho(x_0)$ and $|x - z| \approx |x_0 - z|$, we get

$$\int_{\mathbb{R}^n \setminus 2B} |K_\alpha(x, z) - K_\alpha(y, z)| |b(z) - b_B| |f(z)| dz \lesssim K_1 + K_2$$

where

$$K_1 = r^\delta \int_{Q \setminus 2B} \frac{|f(z)(b(z) - b_B)|}{|x_0 - z|^{n+\delta-\alpha}} dz,$$

and

$$K_2 = r^\delta \rho(x_0)^N \int_{Q^c} \frac{|f(z)(b(z) - b_B)|}{|x_0 - z|^{n+\delta+N-\alpha}} dz.$$

Let j_0 be the smallest integer so that $2^{j_0}r \geq \gamma\rho(x_0)$. Splitting $Q \setminus 2B$ into annuli $2^{k+1}B \setminus 2^k B$ for $k = 1, \dots, j_0$, we obtain

$$K_1 \lesssim \sum_{k=1}^{j_0} \frac{2^{-k\delta}}{|2^k B|^{1-\alpha/n}} \int_{2^k B} |f(z)(b(z) - b_B)| dz.$$

Using Hölder inequality and Proposition 1.3, we obtain

$$K_1 \lesssim \sum_{k=1}^{j_0} k 2^{-k\delta} \|b\|_\theta \left(\frac{1}{|2^k B|^{1-\alpha s/n}} \int_{2^k B} |f(z)|^s dz \right)^{1/s}$$

Note that if $x \in B \cap Q_j$ then $2^k B \subset \tilde{Q}_j$ where Q_j and \tilde{Q}_j are balls in (8). Therefore,

$$K_1 \lesssim \sum_{k=1}^{j_0} k 2^{-k\delta} \|b\|_\theta \inf_{u \in B} \tilde{M}_{\alpha,s} f(u).$$

Splitting Q^c into annuli and then applying Hölder inequality and Proposition 1.3 again, we obtain

$$\begin{aligned} K_2 &\lesssim \left(\frac{\rho(x_0)}{2^{j_0}r} \right)^N \sum_{k=j_0-1}^{\infty} \frac{2^{-k\delta-(k-j_0)N}}{|2^{k-j_0+1}2^{j_0-1}B|^{1-\alpha/n}} \int_{2^k B} |f(z)(b(z) - b_B)| dz \\ &\lesssim \|b\|_\theta \left(\frac{\rho(x_0)}{2^{j_0}r} \right)^N \sum_{k=j_0-1}^{\infty} k 2^{-k\delta-(k-j_0)(N-\theta')} \left(\frac{1}{|2^{k-j_0+1}2^{j_0-1}B|^{1-\alpha s/n}} \int_{2^k B} |f(z)|^s dz \right)^{1/s}. \end{aligned}$$

Since $2^{j_0}r \geq \gamma\rho(x_0) \geq 2^{j_0-1}r$, we get that

$$\begin{aligned} K_2 &\lesssim \|b\|_\theta \sum_{k=j_0-1}^{\infty} 2^{-(k-j_0)(N-\theta')} \left(\frac{1}{|2^{k-j_0+1}Q|^{1-\alpha s/n}} \int_{2^{k-j_0+1}Q} |f(z)|^s dz \right)^{1/s} \\ &\lesssim \|b\|_\theta \sum_{l=1}^{\infty} 2^{-l(N-\theta')} \left(\frac{1}{|2^lQ|^{1-\alpha s/n}} \int_{2^lQ} |f(z)|^s dz \right)^{1/s} \\ &\lesssim \|b\|_\theta \inf_{u \in B} G_{\gamma, \alpha, s} f(u) \text{ since } Q = \gamma B(x_0, \rho(x_0)). \end{aligned}$$

This completes our proof.

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